

A CONNECTION BETWEEN DECOMPOSABILITY OF ULTRAFILTERS AND POSSIBLE COFINALITIES

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ABSTRACT. We introduce the decomposability spectrum $K_D = \{\lambda \geq \omega \mid D \text{ is } \lambda\text{-decomposable}\}$ of an ultrafilter D , and show that Shelah's pcf theory influences the possible values K_D can take.

For example, we show that if \mathfrak{a} is a set of regular cardinals, $\mu \in \text{pcf } \mathfrak{a}$, the ultrafilter D is $|\mathfrak{a}|^+$ -complete and $K_D \subseteq \mathfrak{a}$, then $\mu \in K_D$.

As a consequence, we show that if λ is singular and for some $\lambda' < \lambda$ K_D contains all regular cardinals in $[\lambda', \lambda)$ then:

- (a) if $\text{cf } \lambda = \omega$ then either $\lambda \in K_D$, or $\lambda^+ \in K_D$; and
- (b) if D is $(\text{cf } \lambda)^+$ -complete then $\lambda^+ \in K_D$, and $\text{pp}(\lambda) = \lambda^+$.

1. INTRODUCTION

An ultrafilter D over I is λ -*decomposable* if and only if there is a partition of I into λ sets such that the union of any $< \lambda$ sets of the partition never belongs to D . In other words, D is λ -decomposable if and only if some quotient of D is uniform over λ ; to be more precise, D is λ -decomposable if and only if there exists some ultrafilter D' uniform over λ , and $D' \leq D$ in the *Rudin-Keisler order*. If D is an ultrafilter, define the *decomposability spectrum* K_D of D by $K_D = \{\lambda \geq \omega \mid D \text{ is } \lambda\text{-decomposable}\}$. In this note we address the following question: which are the possible values K_D can take?

It is well known that K_D is closed under taking cofinalities and regular predecessors; more explicitly, if κ is regular and $\kappa^+ \in K_D$ then $\kappa \in K_D$; and, if $\kappa \in K_D$ is singular, then $\text{cf } \kappa \in K_D$. Further constraints on K_D are given in [L]. In this note we show that, under a completeness assumption on D , K_D is closed under Shelah's pcf operation of taking possible cofinalities of reduced products.

We now give a few examples of possible values for K_D . The possibility that K_D is an interval can always occur: if D is uniform over λ and

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(ω, λ) -regular then $K_D = [\omega, \lambda]$. By results from [D] (see also [L]), if there is no inner model with a measurable cardinal then K_D is always an interval with ω as the inferior extreme.

If there is a measurable cardinal μ , then there is a μ -complete ultrafilter D over μ , and this implies $K_D = \{\mu\}$. Conversely, if $|K_D| = 1$, say $K_D = \{\mu\}$, then μ is either ω or a measurable cardinal. Starting from a measurable cardinal, we can create ultrafilters with gaps in their decomposability spectra: as a very elementary example, if D' is non principal over ω , and D is as above, then $K_{D \times D'} = \{\omega, \mu\}$. Of course, we get more interesting examples by destroying the measurability of μ ; for example, by Prikry forcing [P], one can turn the cofinality of μ to ω , thus having an ultrafilter for which $K_D = \{\omega, \mu\}$ and $\text{cf } \mu = \omega$. By a more elaborate forcing, one can obtain $K_D = \{\omega, \mu\}$, for some strongly inaccessible and not weakly compact μ [Shr].

The question of which values K_D may take is intriguing even in case $|K_D| = 2$. This particular question originated from [Si], and not everything is known yet about those μ for which we can have $K_D = \{\omega, \mu\}$. Some restrictions on μ are listed in [L]. Seemingly, the possibility $K_D = \{\lambda, \mu\}$, where λ, μ are both $> \omega$ has never been investigated, apart from trivial cases [L].

However, in the present note we deal with the case when K_D is infinite. In this case things are even more involved, but for a simple reason: suppose that λ is a limit cardinal, $(\lambda_\alpha)_{\alpha \in \text{cf } \lambda}$ is an ascending sequence of cardinals unbounded in λ , and $\lambda_\alpha \in K_D$, for $\alpha \in \text{cf } \lambda$. Without loss of generality, D is uniform, say over μ ; this implies that $\mu \in K_D$, and μ is necessarily larger than all the λ_α 's. The main complication arises from the fact the sequence $(\lambda_\alpha)_{\alpha \in \text{cf } \lambda}$ conveys unclear and confused information about the possible values μ can take.

For example, if we are in the above situation, and D is $(\omega, \text{cf } \lambda)$ -regular, then D is necessarily λ -decomposable [L]; on the other hand, if λ is limit, D is $(\text{cf } \lambda)^+$ -complete, and $\text{pp}(\lambda) = \lambda^+$ then D is λ^+ -decomposable (Proposition 2.5 (d) \Rightarrow (b)). In [L] we asked whether the above possibilities are the only ones which can occur, namely, we asked the following problem.

Problem 1.1. Suppose that λ is a limit cardinal and that there are arbitrarily large cardinals $\lambda' < \lambda$ such that the ultrafilter D is λ' -decomposable.

Is it true that D is either λ -decomposable or λ^+ -decomposable?

In other words, Problem 1.1 asks whether K_D satisfies the following closure property: for every $X \subseteq K_D$ either $\sup X \in K_D$ or $(\sup X)^+ \in K_D$.

In this paper we provide new evidence that Problem 1.1 has an affirmative answer in the great majority of cases when λ is singular. Our results, together with pcf theory, suggest that a negative solution of Problem 1.1 for some singular λ could occur only in very special situations, and would probably exhibit a very complicated structure for K_D . In turn, the existence of an ultrafilter with certain specified values of K_D puts constraints on pcf theory and, in some cases, implies that pcf theory has the simplest possible structure (Theorem 2.3 and Corollary 2.5).

The starting result of the present paper is Theorem 2.1 in the next section. It shows that K_D satisfies the following closure property: if $(\lambda_j)_{j \in J}$ are regular cardinals, D is a $|J|^+$ -complete ultrafilter, and $\lambda_j \in K_D$ for every $j \in J$, then $\mu \in K_D$, for every μ which can be obtained as the cofinality of $\prod_E \lambda_j$ for a suitable ultrafilter E over J . A particular case of Theorem 2.1 is rephrased using the terminology of pcf theory in Corollary 2.2. Then we refine the above mentioned closure property of K_D and show that, in many cases, if $\lambda_j \in K_D$ for every $j \in J$, then $\lambda^+ \in K_D$, where $\lambda = \sup_{j \in J} \lambda_j$. In particular, in Corollary 2.4, we show that Problem 1.1 has an affirmative answer in the particular case in which X is an interval of regular cardinals whose supremum is singular of cofinality ω . In Section 3 we give an alternative proof for some steps in R. Solovay's result that GCH (the Generalized Continuum Hypothesis) holds at strong limit singular cardinals above a strongly compact cardinal. The results in Section 3 do not depend on Section 2, and their proofs do not use pcf theory. At the end, we state some further problems in Section 4. See [L, She] for unexplained notions.

If E is an ultrafilter over J , and $(\lambda_j)_{j \in J}$ are cardinals, we shall write $\text{cf} \prod_E \lambda_i$ to denote the cofinality of the linear order $\prod_E \langle \lambda_i, \leq \rangle$.

If λ is a limit cardinal, the locution “the ultrafilter D is κ -decomposable for all sufficiently large regular $\kappa < \lambda$ ” means that there exists $\lambda' < \lambda$ such that D is κ -decomposable for every regular κ with $\lambda' \leq \kappa < \lambda$. We shall sometimes consider also the weaker condition “there are arbitrarily large (regular) cardinals $\kappa < \lambda$ such that the ultrafilter D is κ -decomposable”, which means that for every $\lambda' < \lambda$ there is some (regular) κ such that $\lambda' \leq \kappa < \lambda$, and D is κ -decomposable.

We shall make use of the following classical result.

Theorem 1.2. [CC, Theorem 1] [KP, Theorem 2.1] *If the ultrafilter D is uniform over λ^+ then D is either cf λ -decomposable, or (λ', λ^+) -regular for some regular $\lambda' \leq \lambda$.*

See [L] for further remarks about Theorem 1.2, as well as for further references and some generalizations.

2. THE DECOMPOSABILITY SPECTRUM AND pcf THEORY

Theorem 2.1. *Suppose that E is an ultrafilter over J , $(\lambda_j)_{j \in J}$ are regular cardinals and $\text{cf} \prod_E \lambda_j = \mu$. If D is a $|J|^+$ -complete ultrafilter and D is λ_j -decomposable for all $j \in J$ then D is μ -decomposable.*

Proof. Since $\text{cf} \prod_E \lambda_j = \mu$, there are functions $g_\alpha \in \prod \lambda_j$ ($\alpha \in \mu$) such that $[g_\alpha]_E$ ($\alpha \in \mu$) is a cofinal sequence in $\prod_E \lambda_i$. Without loss of generality, we can assume $[g_\alpha]_E <_E [g_{\alpha'}]_E$ whenever $\alpha < \alpha' < \mu$.

For every $j \in J$, let f_j be a λ_j -decomposition of D , and suppose that D is over I . Thus, for every $j \in J$ and for every β in λ_j we have $\{i \in I \mid f_j(i) > \beta\} \in D$. In particular, for every $\alpha \in \mu$ and $j \in J$ we have $\{i \in I \mid f_j(i) > g_\alpha(j)\} \in D$. Since D is $|J|^+$ -complete, then for every $\alpha \in \mu$, it happens that $X_\alpha = \bigcap_{j \in J} \{i \in I \mid f_j(i) > g_\alpha(j)\} \in D$.

Thus, if $i \in X_\alpha$ then $f_j(i) > g_\alpha(j)$ for all $j \in J$. Hence, if $i \in X_\alpha$, then $[f_j(i)]_E >_E [g_\alpha]_E$. If we put $Y_\alpha = \{i \in I \mid [f_j(i)]_E >_E [g_\alpha]_E\}$ then $Y_\alpha \supseteq X_\alpha \in D$, hence $Y_\alpha \in D$, for every $\alpha \in \mu$.

Since $[g_\alpha]_E <_E [g_{\alpha'}]_E$ whenever $\alpha < \alpha' < \mu$, we have $Y_\alpha \supseteq Y_{\alpha'}$ whenever $\alpha < \alpha' < \mu$. Moreover, $\bigcap_{\alpha \in \mu} Y_\alpha = \emptyset$, since if, on the contrary, $i \in \bigcap_{\alpha \in \mu} Y_\alpha$, then $[f_j(i)]_E >_E [g_\alpha]_E$ for all $\alpha \in \mu$, and this contradicts the assumption that $[g_\alpha]_E$ ($\alpha \in \mu$) is a cofinal sequence in $\prod_E \lambda_j$.

Thus, we have found a sequence Y_α ($\alpha \in \mu$) of sets in D such that $Y_\alpha \supseteq Y_{\alpha'}$ whenever $\alpha < \alpha' < \mu$, and $\bigcap_{\alpha \in \mu} Y_\alpha = \emptyset$. This means that D is μ -descendingly incomplete. Since μ is a regular cardinal by assumption (being the cofinality of $\prod_E \lambda_i$) we get that D is μ -decomposable. \square

In the above theorem we are not necessarily assuming that all the λ_i 's are distinct. The particular case in which they are all distinct can be, of course, restated in terms of pcf theory [She].

Corollary 2.2. *Suppose that \mathfrak{a} is a set of regular cardinals, $\mu \in \text{pcf } \mathfrak{a}$, D is an $|\mathfrak{a}|^+$ -complete ultrafilter, and D is κ -decomposable for every $\kappa \in \mathfrak{a}$. Then D is μ -decomposable.*

Notice that the condition $|\mathfrak{a}| < \min \mathfrak{a}$ follows from the hypotheses of Corollary 2.2, since if the ultrafilter D is $|\mathfrak{a}|^+$ -complete and κ -decomposable then necessarily $\kappa > |\mathfrak{a}|$.

We now show that, for λ singular, if $K_D \cap \lambda$ is an interval of regular cardinals cofinal in λ , and D is $(\text{cf } \lambda)^+$ -complete, then $\lambda^+ \in K_D$. In turn, λ^+ -decomposability, together with $(\text{cf } \lambda)^+$ -completeness, implies $\text{pp}(\lambda) = \lambda^+$. This means that, in some sense, the pcf theory at λ has the simplest possible structure, and, as argued in [She], it is a version of the Generalized Continuum Hypothesis. Moreover, $\text{pp}(\lambda) = \lambda^+$ and Corollary 2.2 imply that we can equivalently suppose that there

are arbitrarily large regular $\kappa < \lambda$ such that D is κ -decomposable, as exemplified in Proposition 2.5 below. A pcf-free version of Theorem 2.3 will be given in Proposition 3.1.

Theorem 2.3. *Suppose that λ is a singular cardinal, and the ultrafilter D is $(\text{cf } \lambda)^+$ -complete and κ -decomposable for all sufficiently large regular $\kappa < \lambda$.*

Then D is λ^+ -decomposable, and moreover $\text{pp}(\lambda) = \lambda^+$.

Proof. By [She, II, Theorem 1.5] there is a strictly increasing sequence λ_α ($\alpha \in \text{cf } \lambda$) of regular cardinals with $\lambda_\alpha < \lambda$, and $\sup_{\alpha \in \text{cf } \lambda} \lambda_\alpha = \lambda$, and there is a uniform ultrafilter E on $\text{cf } \lambda$ such that $\text{cf } \prod_E \lambda_\alpha = \lambda^+$. Actually, [She, II, Theorem 1.5] obtains the above result for the ideal $J_{\text{cf } \lambda}^{\text{bd}}$ in place of the ultrafilter E ; however, it is enough to take as E any ultrafilter extending the dual of $J_{\text{cf } \lambda}^{\text{bd}}$ (cf. e. g. the proof of [BM, Lemma 1.4]). Notice that the fact that $J_{\text{cf } \lambda}^{\text{bd}}$ is the ideal of sets bounded in $\text{cf } \lambda$ implies that E is uniform over $\text{cf } \lambda$.

By assumption, there is $\lambda' < \lambda$ such that D is κ -decomposable for every regular κ with $\lambda' \leq \kappa < \lambda$. Since E is uniform, the set $X = \{\alpha \in \text{cf } \lambda \mid \lambda' \leq \lambda_\alpha\}$ belongs to E . Hence, if E' is the restriction of E to X , the hypotheses of Theorem 2.1 apply with E' in place of E , and we get that D is λ^+ -decomposable.

We now show that if there is a λ^+ -decomposable and $(\text{cf } \lambda)^+$ -complete ultrafilter D then $\text{pp}(\lambda) = \lambda^+$. Without loss of generality, we can suppose that D is uniform on λ^+ . Since D is $(\text{cf } \lambda)^+$ -complete, D is not $\text{cf } \lambda$ -decomposable, hence, by Theorem 1.2, there is $\lambda' < \lambda$ such that D is (λ', λ^+) -regular. Hence, D is κ -decomposable for all regular κ 's with $\lambda' \leq \kappa < \lambda$.

Suppose by contradiction that $\text{pp}(\lambda) > \lambda^+$. By the “No Hole Conclusion” [She, II, 2.3(1)], $\lambda^{++} \in \text{pcf } \mathfrak{a}$, where \mathfrak{a} is a set of regular cardinals cofinal in λ , and $|\mathfrak{a}| = \text{cf } \lambda$ (again [She] deals with an ideal, but it is sufficient to extend the dual of this ideal to an ultrafilter, which turns out to be uniform over $\text{cf } \lambda$). By considering, as above, a final segment of \mathfrak{a} , we get from Corollary 2.2 that D is λ^{++} -decomposable, but this is impossible, since D is uniform over λ^+ . \square

Corollary 2.4. *If λ is a singular cardinal of cofinality ω , and the ultrafilter D is κ -decomposable for all sufficiently large regular $\kappa < \lambda$ then D is either λ -decomposable, or λ^+ -decomposable.*

Proof. If D is ω_1 -complete, then D is λ^+ -decomposable, by Theorem 2.3.

On the other side, if D is not ω_1 -complete, then D is ω -decomposable, hence λ -decomposable by [P1, Proposition 1, and footnote on p. 461]. See also [L] for generalizations of results from [P1]. \square

Proposition 2.5. *If λ is a singular cardinal and the ultrafilter D is $(\text{cf } \lambda)^+$ -complete, then the following conditions are equivalent:*

- (a) *D is κ -decomposable for all sufficiently large regular $\kappa < \lambda$;*
- (b) *D is λ^+ -decomposable;*
- (c) *There is some $\lambda' < \lambda$ such that D is (λ', λ^+) -regular;*
- (c') *There is some $\lambda' < \lambda$ such that D is (λ', λ) -regular;*
- (d) *$\text{pp}(\lambda) = \lambda^+$ and there are arbitrarily large regular cardinals $\kappa < \lambda$ such that D' is κ -decomposable.*

Proof. (a) \Rightarrow (b) \Rightarrow (c) and (b) \Rightarrow (d) are given by the proof of Theorem 2.3.

(d) \Rightarrow (b) follows from Corollary 2.2, and (c) \Rightarrow (c') \Rightarrow (a) are trivial. \square

3. A GENERALIZATION OF SOLOVAY'S GCH RESULT

Some arguments from the proof of Theorem 2.3 (with no use of pcf theory) can be used to furnish an alternative proof of R. Solovay's GCH result.

Proposition 3.1. *Suppose that λ is a singular cardinal, $\text{cf } \lambda < \kappa$, and $\nu^{<\kappa} < \lambda$ for every $\nu < \lambda$. If there exists an ultrafilter D which is λ^+ -decomposable and κ -complete, then $(\lambda^+)^{<\kappa} = \lambda^+$.*

Proof. Without loss of generality, we can suppose that D is uniform over λ^+ . Since D is λ^+ -decomposable and not $\text{cf } \lambda$ -decomposable (being κ -complete and $\kappa > \text{cf } \lambda$), then, by Theorem 1.2, D is (ν, λ^+) -regular for some $\nu < \lambda$.

Since D is κ -complete then, by a remark which is probably due to Solovay (see [L]), D is $((\nu^{<\kappa})^+, (\lambda^+)^{<\kappa})$ -regular. Since $\nu^{<\kappa} < \lambda$, and D is uniform over λ^+ , this can happen only if $(\lambda^+)^{<\kappa} = \lambda^+$. \square

Theorem 3.2. [So] *Suppose that κ is μ -strongly compact.*

- (a) *If $\kappa \leq \nu \leq \mu$, and ν is regular, then $\nu^{<\kappa} = \nu$.*
- (b) *If $\kappa \leq \lambda \leq \mu$, and λ is singular and strong limit, then $2^\lambda = \lambda^+$.*

Proof. That κ is μ -strongly compact implies that there is a κ -complete (κ, μ) -regular ultrafilter. Hence, for every regular ν with $\kappa \leq \nu \leq \mu$ there is a κ -complete ν -decomposable ultrafilter.

(a) is now proved by induction on ν . Since κ is measurable, hence strongly inaccessible, and because of standard cardinal arithmetic, the only non trivial case is when ν is the successor of a singular cardinal

λ of cofinality $< \kappa$. In this case, we can apply Proposition 3.1 because of the above remark.

(b) In the case when $\text{cf } \lambda < \kappa$, case (b) follows from (a) by standard cardinal arithmetic. The case $\text{cf } \lambda \geq \kappa$ can be obtained as a consequence of Silver's Theorem (see [KM, p. 191]) from the case $\text{cf } \lambda < \kappa$, or, alternatively, using the appropriate arguments from [So]. \square

4. SOME PROBLEMS

Problem 4.1. If we assume that E is uniform, and that $|J| < \lambda_i$ for all i , can the hypothesis “ D is $|J|^+$ -complete” in Theorem 2.1 be weakened to “ D is not $|J|$ -decomposable”?

The assumption “ $|J| < \lambda_i$ for all i ” in Problem 4.1 is necessary, otherwise there are easy counterexamples. However, the assumption that D is $(\text{cf } \lambda)^+$ -complete can be weakened to “ D is not $\text{cf } \lambda$ -decomposable”, in Theorem 2.3 and Proposition 2.5.

Proposition 4.2. *If λ is a singular cardinal and the ultrafilter D is not $\text{cf } \lambda$ -decomposable, then the following conditions are equivalent:*

- (a) *D is κ -decomposable for all sufficiently large regular $\kappa < \lambda$;*
- (b) *D is λ^+ -decomposable;*
- (c) *There is some $\lambda' < \lambda$ such that D is (λ', λ^+) -regular;*
- (c') *There is some $\lambda' < \lambda$ such that D is (λ', λ) -regular;*
- (d) *$\text{pp}(\lambda) = \lambda^+$ and there are arbitrarily large regular cardinals $\kappa < \lambda$ such that D' is κ -decomposable.*

A major problem is what happens if in the hypothesis of Theorem 2.3, Corollary 2.4 and Proposition 2.5(a) we only suppose that there are arbitrarily large regular cardinals $\kappa < \lambda$ such that D is κ -decomposable. We have partial results showing that the statements still hold, except possibly for very special situations.

We can ask even a subtler problem. Does the first conclusion in Theorem 2.3 hold when decomposability is replaced by regularity? Namely, is the following true?

Conjecture 4.3. Suppose that λ is a singular cardinal, $\mu < \lambda$ and the ultrafilter D is $(\text{cf } \lambda)^+$ -complete and (μ, κ) -regular for all $\kappa < \lambda$.

Then D is (μ, λ^+) -regular.

Again, we expect to get this, except perhaps for really special situations, and it is likely that the assumption that D is $(\text{cf } \lambda)^+$ -complete can be weakened to “ D is not $\text{cf } \lambda$ -decomposable”.

A positive solution to the above problems would furnish a solution to a lot of problems raised in [L].

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